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The Effect of a Longitudinal Gravitational Field on the Super-cavitating Flow Over a Wedge

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The free-streamline flow past a symmetrical wedge in the presence of a longitudinal gravitational field is determined with a linearized theory. The proportions of the cavity depend upon the cavitation number and Froude number. The drag coefficient is likewise affected by gravity, though to a smaller extent.

In recent years a number of papers treating linearized free-streamline problems have appeared subsequent to Tulin's introductory paper on this subject [1].[1] Among these may be mentioned Wu's extension of Tulin's method for supercavitating hydrofoils with arbitrary shape and cavitation number [2], hydrofoils with cavitation only near the leading edge [3], supercavitating hydrofoils in cascade [4], and Cohen's work on wall-interference effects [5]. In all of these works the hydrofoil is assumed to be in a force-free field. However, Parkin recently has estimated the effect of a gravity field normal to the direction of the flow by means of a simplified representation of the gravity effect on the cavity boundary condition [6]. As yet the problem of the longitudinal gravity field does not seem to have been discussed.

[1] Numbers in brackets designate References at the end of the paper.

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Fully cavitated flows are known to occur in axial gravitational fields. The cavity associated with vertical water entry or exit is one example. An effect similar to that of axial gravity occurs when fully cavitating flow takes place in a large water tunnel with slightly diverging walls. The longitudinal pressure gradient that results from the variable cross section plays a role much like that of a force field. It appears then, that to have an understanding of free-streamline problems in all cases of possible technical interest, the effect of an axial or longitudinal gravitational field must be examined.

Formulation of Problem

A sketch of the cavitating wedge is shown in Fig. 1 where for convenience all lengths are made dimensionless by dividing by the length of the wedge L. The central idea of thin-airfoil theory is the linearization of the surface boundary conditions. The analysis is further simplified by fulfilling these conditions on the axis rather than on an approximate neighboring shape. In linearized free-streamline theory both of these simplifications are made. In the spirit of these approximations we will require that the velocity never differ too much from the free-stream velocity and, further, that the slope of the body must be small.

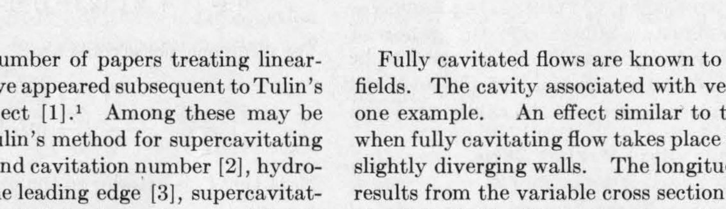

Fig. 1 Sketch of cavity configuration

Nomenclature

a, b, A, B = constants

C_D = drag coefficient $\left(D \Big/ \frac{\rho}{2} U^2 L\right)$

C_p = pressure coefficient $= (p - p_c)/\rho U^2/2$

D = drag force

F^2 = Froude number squared (U^2/gL)

g = gravitational acceleration

$i = \sqrt{-1}$

K = cavitation number $[(p_0 - p_c)/\rho U^2/2]$

L = length of wedge

l = cavity length divided by L

p = pressure

q_c = characteristic velocity on the cavity $[U(1 + K)^{1/2}]$

S = dimensionless cavity cross-sectional area

U = velocity at infinity

u = perturbation velocity in direction of U

v = perturbation velocity normal to U

$w = u - iv$

x, y = Cartesian co-ordinates in physical plane

$z = x + iy$

γ = wedge, semi-apex angle

ρ = fluid density

ζ = transformed plane

$(\ )_c$ = quantity on cavity

$(\ )_0$ = quantities far from the body but in plane of base of wedge

$\overline{(\ )}$ = the complex conjugate

In the thin-airfoil theory, the characteristic velocity is the velocity at infinity. However, it was shown by Wu [2] that a better approximation to the nonlinear theory was obtained if the velocity on the cavity was used as the characteristic velocity. In the case presently to be considered, the fluid velocity along the cavity is not constant owing to the effect of gravity. For the purpose of definiteness, the fluid velocity at the base of the wedge is used as the characteristic velocity. The connection between the characteristic velocity and the free-stream velocity is provided by the Bernoulli equation

$$p_0 + \frac{\rho U^2}{2} + \rho g = p_c + \frac{\rho}{2} q_c^2 + \rho g = p + \frac{\rho}{2} q^2 + \rho g x \quad (1)$$

The reference datum for the longitudinal gravity field is taken at $x = 0$. Thus if $K = (p_0 - p_c)/\rho U^2/2$, $q_c = U\ (1 + K)^{1/2}$. The velocity vector q is

$$q = q_c + u + iv \quad (2)$$

where u, v are perturbation velocity components assumed to be much smaller than q_c. We now define a pressure coefficient based upon the free-stream dynamic pressure and the cavity pressure:

$$C_p = \frac{p - p_c}{\rho U^2/2} = \frac{2g(1 - x)}{U^2} - \frac{2uq_c}{U^2} \quad (3)$$

where u^2, v^2 have been neglected compared to q_c^2. It should be noted from equation (1) that the gravity force points upstream, Fig. 1.

As in other thin-airfoil theories, the boundary conditions are applied on the chord line of the profile which in the present case is the slit of length l on the real z-axis. The formulation of the problem can now be completed by stating the boundary conditions. They are:

(*a*) $v = \pm q_c\gamma$ on the wedge $(0 \leq x \leq 1)$,

(*b*) $C_p = 0$ on the cavity or from equation (3)

$$\frac{u}{q_c} = \frac{g(1 - x)}{q_c^2} \quad \text{for } 1 \leq x \leq l$$

(*c*) As $z \to \infty$, $q_c + u \to U$, $v \to 0$. Thus

$$\frac{u}{q_c} = \frac{1}{(1 + K)^{1/2}} - 1 \quad \text{far from the body} \quad (4)$$

(*d*) The combination of body and cavity must be closed. The equivalent statement is that the net source strength must be zero.

(*e*) Lastly, the flow cannot contain nonintegrable singularities on the slit or have multiple values off the slit.

Conditions (*a*) – (*e*) are sufficient to determine the flow field although it has not been proved that the solution is unique.[2]

[2] Other linearized models of the flow are possible. For example, see Cohen's linearized model of the notched hodograph [5]. However, for the present purpose Tulin's original model seems simplest.

Solution

The complex velocity $w = u - iv$ is an analytic function of $z = x + iy$. It can therefore be transformed to other more convenient planes in such a way that w is the same at corresponding points. The plane chosen for analysis is the semicircular plane shown in Fig. 2. It is the same as that used by Wu [7] except that the physical plane is transformed onto the upper-half ζ-plane. The appropriate boundary conditions are also shown in Fig. 2.

It can be verified that the mapping function that transforms the z-plane onto the upper half ζ-plane is

$$z = l\left[1 - \frac{4(l - 1)\zeta^2}{(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)}\right] \quad (5)$$

where ζ_1, ζ_2 are the roots of

$$0 = \zeta^4 + 2\zeta^2(2l - 1) + 1 = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2) \quad (6)$$

For reference, these roots are

$$\begin{aligned} \zeta_1 &= i\,[l^{1/2} + (l - 1)^{1/2}] \\ \zeta_2 &= i\,[l^{1/2} - (l - 1)^{1/2}] \end{aligned} \quad (7)$$

Note that ζ_1 is exterior to the unit circle and represents the point $z = \infty$.

The solution proceeds as follows: Separate w into two terms that satisfy the following conditions:

For w_1

$$\begin{aligned} v_1 &= \pm q_c\gamma \quad \text{on the wedge,} \\ u_1 &= g/q_c \quad \text{on the cavity} \end{aligned} \quad (4a)$$

For w_2

$$\begin{aligned} v_2 &= 0 \text{ on the wedge} \\ u_2 &= -\frac{gx}{q_c} \quad \text{on the cavity} \end{aligned} \quad (4b)$$

The sum of $w_1 + w_2$ must satisfy the remainder of conditions (4). The solution of w_1 subject to the foregoing restrictions is

$$w_1 = -\frac{2\gamma}{\pi} q_c \ln \frac{\zeta + i}{\zeta - i} + iA\left(\zeta - \frac{1}{\zeta}\right) + B \quad (8)$$

where A, B are real constants. From (4*a*), $B = g/q_c$ but A cannot yet be determined.

Term w_2 poses somewhat more of a problem. It can be seen from equation (5) that $x(\zeta)$ is a complicated function. However, a simple closed-form solution for w_2 can be obtained by use of the transformation function, equation (5), and suitable images. Such a possibility is immediately suggested since on the real ζ-axis (i.e., on the cavity) equation (5) is equal to x. Hence we might suspect that $w_2(\zeta) \sim z$. However, $z(\zeta)$ cannot be directly used as the velocity function since it has a pole at ζ_1 (or $z = \infty$). (ζ_2 is interior to the unit circle and therefore does not represent a point in

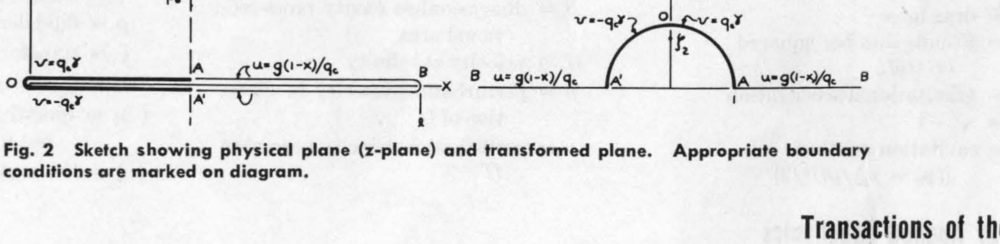

Fig. 2 Sketch showing physical plane (z-plane) and transformed plane. Appropriate boundary conditions are marked on diagram.

the physical plane.) What must be done, therefore, is to remove the singularity at ζ_1 and add suitable images to make v_2 zero on the unit circle. At the same time the contribution of the images on the real (ζ) axis must be purely imaginary to maintain the property $u_2 \sim x$ there. With these preliminaries it can be seen that

$$w_2 = -\frac{gl}{q_c}\left[1 - \frac{4(l-1)\zeta^2}{(\zeta^2-\zeta_1^2)(\zeta^2-\zeta_2^2)} + \frac{2(l-1)\zeta_1}{(\zeta-\zeta_1)(\zeta_1^2-\zeta_2^2)} + \frac{2(l-1)\bar{\zeta}_1}{\left(\frac{1}{\zeta}-\bar{\zeta}_1\right)(\bar{\zeta}_1^2-\bar{\zeta}_2^2)} - \frac{2(l-1)\bar{\zeta}_1}{(\zeta-\bar{\zeta}_1)(\bar{\zeta}_1^2-\bar{\zeta}_2^2)} - \frac{2(l-1)\zeta_1}{\left(\frac{1}{\zeta}-\zeta_1\right)(\zeta_1^2-\zeta_2^2)}\right]$$

obeys the requirements of equation (4*b*). With the aid of equation (7) this expression is capable of considerable simplification and becomes, after some manipulation,

$$w_2 = -\frac{gl}{q_c}\left\{1 - \left(\frac{l-1}{l}\right)^{1/2}\left[\frac{\zeta_1}{\zeta+\zeta_1} - \frac{\zeta_2}{\zeta+\zeta_2}\right]\right\} \tag{9}$$

The complete velocity function is $w_1 + w_2$ or

$$\frac{w}{q_c} = \frac{w_1+w_2}{q_c} = -\frac{2\gamma}{\pi}\ln\frac{\zeta+i}{\zeta-i} + i\frac{A}{q_c}\left(\zeta - \frac{1}{\zeta}\right) - \frac{g}{q_c^2}(l-1) + \frac{gl}{q_c^2}\left(\frac{l-1}{l}\right)^{1/2}\left[\frac{\zeta_1}{\zeta+\zeta_1} - \frac{\zeta_2}{\zeta+\zeta_2}\right] \tag{10}$$

We have yet to determine A and to find the relation between γ, K, and g. However, conditions (c) and (d) of (4) remain to be fulfilled. These operations can be carried out in the ζ-plane, but as pointed out in [7], it is easier to work in the z-plane for this purpose. We require according to this method the expansion of $w(z)$ for large z in the form

$$w(z) = a_0 + \frac{a_1 + ib_1}{z} + \frac{a_2 + ib_2}{z^2} + \ldots \tag{11}$$

Then from (c) of equation (4)

$$a_0 = \frac{1}{(1+K)^{1/2}} - 1 \tag{12}$$

and from (d)

$$a_1 = 0 \tag{13}$$

As $z \rightarrow \infty$, $\zeta \rightarrow \infty$ also. ζ can be found in terms of $1/z$ from equation (5) with the result

$$\zeta = \zeta_1\left[1 + \frac{[l(l-1)]^{1/2}}{2z} + \frac{[l(l-l)]^{1/2}}{8z^2}\{2l + 1 + [l(l-1)]^{1/2}\} + 0\left(\frac{1}{z^3}\right)\right] \tag{14}$$

Substitution of (14) into (10) and simplification gives the desired form

$$\frac{w}{q_c} = -\frac{\gamma}{\pi}\ln\frac{l^{1/2}+1}{l^{1/2}-1} - \frac{A}{q_c}2l^{1/2} - \frac{g}{2q_c^2}(l-1) + \frac{1}{z}\left[\frac{\gamma}{\pi}l^{1/2} - \frac{A}{q_c}(l-1)l^{1/2} - \frac{g}{8q_c^2}(l-1)^2\right] + \frac{1}{z^2}\left[\frac{\gamma}{4\pi}(l+1)l^{1/2} - \frac{A}{4q_c}(l-1)(3l+1)l^{1/2} - \frac{gl}{16q_{c2}}(l-1)(l^2-1)\right] + \ldots \tag{15}$$

The constant A can now be found as indicated by equation (13) and is

$$A = \frac{\gamma q_c}{\pi(l-1)}\left[1 - \frac{\pi g(l-1)^2}{8\gamma q_c^2 l^{1/2}}\right] \tag{16}$$

Equations (15) and (16) constitute the solution. In the next section the principal results are summarized.

Results

We obtain first the relation between the cavity length l, the cavitation number K, and the gravity effect. Let $F^2 = U^2/g$ be the square of the Froude number (recall that the length of the wedge is unity—in the general case $F^2 = U^2/gL$). With equations (16) and (13) we get

$$1 - \frac{1}{(1+K)^{1/2}} - \frac{l-1}{4F^2(1+K)} = \frac{\gamma}{\pi}\left[\ln\frac{l^{1/2}+1}{l^{1/2}-1} + \frac{2l^{1/2}}{l-1}\right] \tag{17}$$

The cavity area (for unit wedge length) is shown in [7] to be $S = -2\pi a_2$. It then follows from equations (15) and (16) that the cavity area is

$$S = \gamma\, l^{3/2} - \frac{\pi}{16F^2}\frac{(l-1)^3}{1+K} \tag{18}$$

The drag coefficient of the body can be expressed as

$$C_D = D\Big/\frac{\rho}{2}U^2L = -\oint_{\text{body}} C_p dy = -\oint_{\text{body}}\left[\frac{(2g(1-x)}{U^2} - \frac{2uq_c}{U^2}\right]dy \tag{19}$$

in which the expression for the pressure coefficient equation (3) has been used. Now $dy = dx\, v/q_c$ and furthermore $2uv = -\text{Im}(w^2)$ so that equation (19) can be written

$$C_D = -\oint_{\text{body}}\frac{2g(1-x)}{U^2}dy - \frac{1}{U^2}\text{Im}\oint_{\text{body}} w^2 dz \tag{20}$$

since on the wetted portion of the body $dz = dx$ (Im denotes the imaginary part). The contour around the body can be considered a part of a contour H that encloses the body, cavity, and the cavity closure denoted by ϵ, Fig. 3. The second term of equation

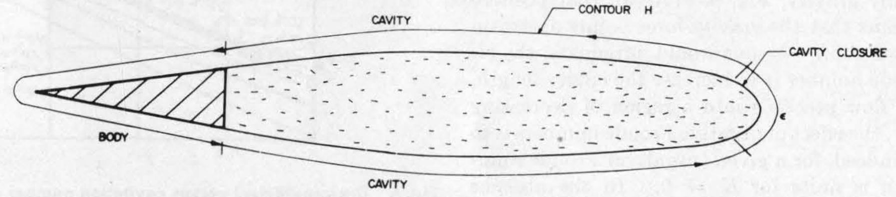

Fig. 3 Definition sketch of contour used in evaluation of drag integral

(20) can now be transformed further by deforming contour H until it consists of a circle of large radius. It is seen immediately from equations (11), (13) that w^2 has no simple pole within H, hence

$$0 = \oint_H w^2 dz = \oint_{\text{body}} w^2 dz + \oint_{\text{cavity}} w^2 dz + \oint_{\epsilon} w^2 dz$$

or

$$-\text{Im} \oint_{\text{body}} w^2 dz = \text{Im} \oint_{\epsilon} w^2 dz + \text{Im} \oint_{\text{cavity}} w^2 dz$$

$$= \text{Im} \oint_{\epsilon} w^2 dz - \oint_{\text{cavity}} 2g(1 - x) \frac{v}{q_c} dx$$

where boundary condition (4b) has been used in the second of these integrals. The expression for the drag coefficient now becomes

$$C_D = - \oint_{\text{body} + \text{cavity}} \frac{2g(1 - x)}{U^2} dy + \frac{1}{U^2} \text{Im} \oint_{\epsilon} w^2 dz$$

We only need to observe that

$$\oint dy = 0$$

is precisely the closure condition and that

$$\oint x \, dy = S,$$

thus

$$C_D = \frac{2S}{F^2} + \frac{1}{U^2} \text{Im} \oint_{\epsilon} w^2 dz \tag{21}$$

Apart from its effect on w, the gravity field gives rise to a buoyant force equal to the product of the area S and the specific weight of the fluid. Indeed, this contribution to the drag coefficient could have been written down without calculation. The usefulness of equation (21) is in the fact that the velocity function w has a particularly simple expansion about the point $z = l$. In fact as

$$z \to l, \zeta \to 2i(l - 1)^{1/2} \Big/ \left(\frac{z}{l} - 1\right)^{1/2} \tag{22}$$

the velocity function, equation (10), becomes

$$w(z) \to - 2A(l - 1)^{1/2} \Big/ \left(\frac{z}{l} - 1\right)^{1/2} \tag{23}$$

Straightforward application of equation (21) gives the final result

$$C_D = \frac{8\gamma^2(1 + K)}{\pi} \left(\frac{l}{l - 1}\right) + \frac{2\gamma l^{1/2}}{F^2} \tag{24}$$

Discussion

The expressions for cavity length, area, and drag coefficient were calculated as a function of cavitation number and Froude number for a wedge with a 15-deg semiapex angle. These results are shown in the graphs of Figs. 4, 5, 6. The cavity length is strikingly affected by gravity, Fig. 4. (Recall that positive F^2 or gravity effect means that the gravity force points upstream and negative F^2 downstream.) As one would anticipate, the effect of a positive Froude number is to increase the cavity length, since in this case the flow proceeds into a region of decreasing pressure. Conversely, the effect of negative Froude numbers is to shorten the cavity. Indeed, for a given (negative) Froude number, the cavity length is finite for $K = 0$. In the absence of gravity ($F^2 = \infty$), the cavity length is infinite for $K = 0$. On

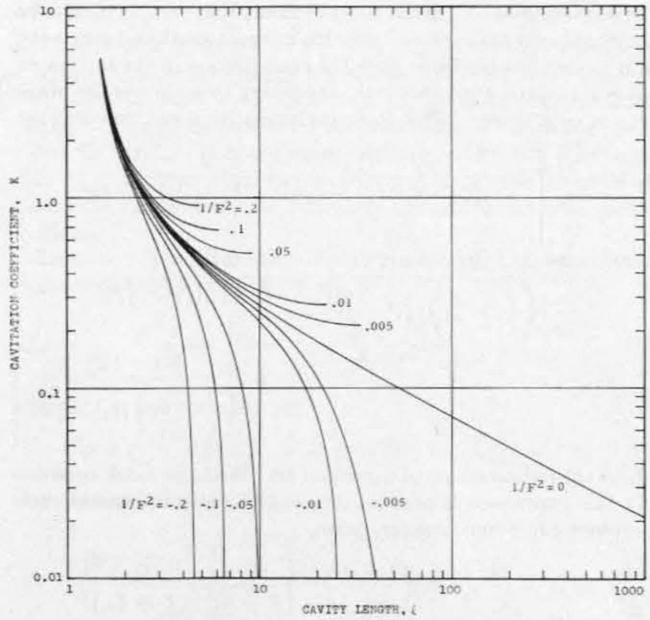

Fig. 4 Cavity length versus cavitation number for various values of Froude number. The semiwedge angle is 15 deg (γ = 15 deg).

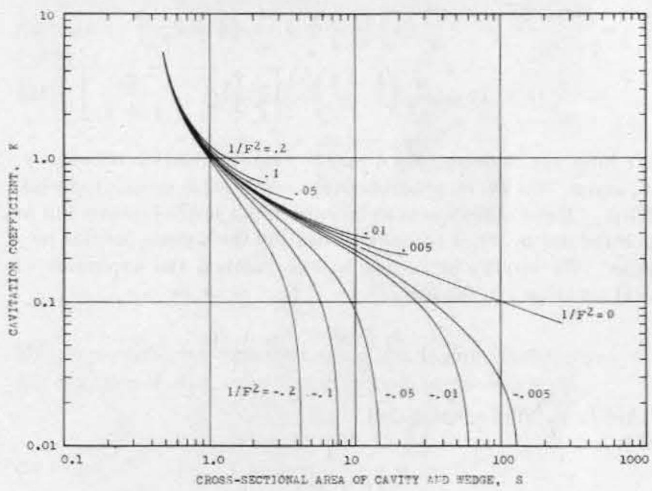

Fig. 5 Dimensionless cavity area versus cavitation number as a function of Froude number for γ = 15 deg

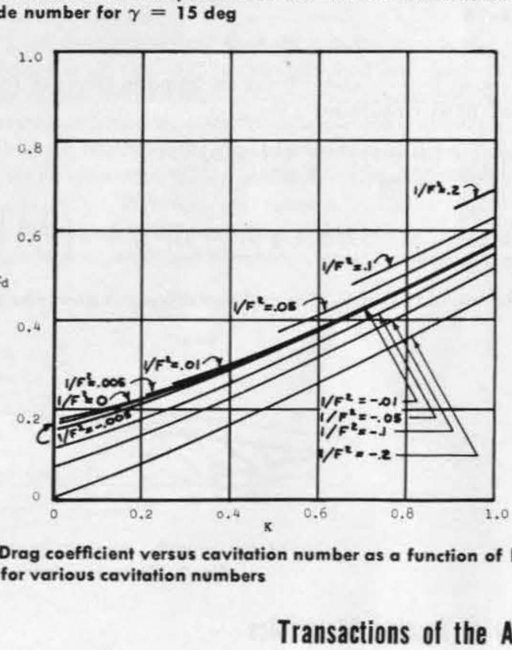

Fig. 6 Drag coefficient versus cavitation number as a function of Froude number for various cavitation numbers

the other hand, for a given positive Froude number, there is a minimum value of the cavitation number. In Fig. 4 the curves of l versus K are not extended beyond this minimum.

The effect of gravity on the drag coefficient, Fig. 6, is somewhat less important than on the length or area. The trend is again as one would expect from physical grounds; namely, that when the force field points upstream, the drag force is increased. Negative values of $\mathbf{F}^2$ decrease the drag force for the same reason. These effects may be partly likened to a horizontal buoyant force as inspection of equation (21) shows. From an inspection of Fig. 5 it appears that the buoyant terms could actually give rise to a negative drag coefficient for a sufficiently strong downstream gravity vector. Such a condition is not likely to occur, however, in practice.

Acknowledgments

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ERRATUM

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Dr. B. R. Parkin of RAND Corporation has pointed out to me that the last bracket in Eq. (24) should be squared. The resulting formula for the drag coefficient then simplifies to

$$C_D = \frac{8\gamma^2 (1+K)}{\pi} \frac{\ell}{\ell - 1} + \frac{2\gamma \sqrt{\ell}}{F^2} \quad .$$

The plot of drag coefficient vs. cavitation number (Fig. 6 of the report) is correct however.